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A Lattice Characterization of Convexity

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ABSTRACT

The lattice of convex subsets of a real vector space is characterized, and conditions for modularity in the lattice are obtained.

The following axioms characterize the lattice L of convex sets in a vector space over an ordered skew field. By adding, two further axioms we obtain the reals as our coordinatizing field, and answer a problem in Birkhoff [2].

We adopt the convention that all lower case Latin letters with or without subscripts with the exception of r, s, t, and u will represent atoms of L, while r, s, t and u will represent arbitrary elements of L.

All terms used are ordinary lattice theoretic terms with the following exceptions:

DEFINITION. $a \vee b$ and $c \vee d$ are said to be *skew* if $\forall\ e, f, g, h$ such that $a \vee b \leqslant e \vee f$ and $c \vee d \leqslant g \vee h$, then $(e \vee f) \wedge (g \vee h) = 0$.

DEFINITION. $a \vee b$ and $c \vee d$ are said to be *parallel* iff $a \vee b$ and $c \vee d$ are skew and

$$[(a \vee c) \wedge (b \vee d)] \vee [(a \vee d) \wedge (b \vee c)] \neq 0.$$

Let L be a complete atomic upper-continuous lattice with join dense atoms satisfying the following axioms:

(1) $[c, d \leqslant a \vee b, c \neq d] \Rightarrow [c \leqslant a \vee d$ iff $c \nleqslant b \vee d]$.

(2) $[c \leqslant a \vee e, d \leqslant b \vee e] \Rightarrow [(b \vee c) \wedge (a \vee d) \neq 0]$.

(3) $[a \leqslant x_1 \vee \cdots \vee x_n] \Rightarrow [\exists\, b \leqslant x_1 \vee \cdots \vee x_{n-1}$, such that $a \leqslant b \vee x_n]$.

(4) [$\forall\, a, b\ \exists\, c$ such that $b \leqslant a \vee c$ but $c \nleqslant a \vee b$].

(5) $\exists\, a, b, c$ and d such that $(a \vee b)$ and $(c \vee d)$ are skew but not parallel.

(6) [$c \wedge (a \vee b) = 0, a \wedge (b \vee c) = 0$ and $b \wedge (a \vee c) = 0$] $\Rightarrow$ [$\exists\, d$ such that $(c \vee d) \parallel (a \vee b)$; and if $(c \vee e) \parallel (a \vee b)$ $\exists\, f, g$ such that $(c \vee d \vee e) \leqslant (f \vee g)$].

We think of the atoms of L as points and the joins of two distinct atoms as segments. We define $\langle ab \rangle = \{c \mid c \leqslant a \vee b\}$. By using axiom 3 and induction we can show that, if Y is a set of atoms, then $d \leqslant \bigvee Y$ implies there exists Y_d (finite) $\subseteq Y$ such that $d \leqslant \bigvee Y_d$. Furthermore, if Y is a set of atoms such that $a, b \in Y \Rightarrow \langle ab \rangle \subseteq Y$, then $d \leqslant \bigvee Y \Rightarrow d \in Y$. Thus we have a correspondence between the lattice elements and certain sets of atoms whose characterization coincides with our intuitive notion of convexity.

For $a, b \in L$, $a \neq b$, we define

$$l(ab) = \{c \mid c \leqslant a \vee b, a \leqslant b \vee c \text{ or } b \leqslant a \vee c\}.$$

We define $l(aa)$ to be a. It is clear that $l(ab) \in L$ for all a and b. Prenowitz in [5] has defined a dependence relation and closed elements in a lattice using this dependence. We repeat his definitions here and show that his closed elements in our convex lattice are exactly our affine ones.

DEFINITION. We say r is *dependent* on $\{x_1, ..., x_n\}$ in L iff $\exists$ an arrangement of the x_i such that

$$(x_{i_1} \vee \cdots \vee x_{i_n}) \wedge (x_{j_1} \vee \cdots \vee x_{j_n}) = 0$$

but

$$(r \vee x_{i_1} \vee \cdots \vee x_{i_n}) \wedge (x_{j_1} \vee \cdots \vee x_{j_n}) \neq 0.$$

DEFINITION. We say that $r \in L$ is *closed in the sense of Prenowitz* iff whenever p is dependent on $\{p_1, ..., p_n\}$ with $p_i \leqslant r$ then $p \leqslant r$.

We can now prove that an element $r \in L$ is closed in the sense of Prenowitz iff whenever $x, y \leqslant r$ then $l(x, y) \leqslant r$. We wish to call such elements *affine*.

We can now prove that, for $r \in L$, r is affine iff $M(s, r)$ for all $s \in L$. (This proof follows along the lines of a proof in Prenowitz.) One can give examples to show that the affine elements do not form a sublattice of L. However, they do form a meet-sublattice of L.

We can easily verify the expected properties of a line, i.e., if x and y are distinct with $x, y \leqslant l(ab)$ then $l(xy) = l(ab)$, and if $l(xy) \neq l(ab)$ then $l(xy) \wedge l(ab) =$ an atom or 0.

We can show that, for $r, s \in L$ with $r \wedge s = 0$ and x in L such that

$x \wedge r = 0$ and $x \wedge s = 0$, then either $(r \vee x) \wedge s = 0$ or $(s \vee x) \wedge r = 0$. This is the lattice theoretic form of the Kakutani lemma. By using a Zorn lemma argument one can show Stone's theorem, i.e., that if $r \wedge s = 0$ then $\exists\, r \leqslant t$, $s \leqslant u$ such that $t \wedge u = 0$ and $t \vee u = 1$.

If a, b, and c are such that $a \nleqslant l(bc)$, then we can think of $a \vee b \vee c$ intuitively as a triangle, and we can show that $l(ab) \vee l(bc)$ behaves like a plane, i.e., the smallest affine set containing a, b, and c. We show that axiom 6 is the ordinary parallel axiom, and define an equivalence relation on the atoms of our lattice in a "plane" $l(ab) \vee l(ac)$ by stating that, for a line $l(a, b)$,

$$e \underset{l(ab)}{\sim} f \qquad \text{iff} \qquad e \vee f \leqslant l(ab) \text{ or } (e \vee f) \wedge l(ab) = 0.$$

We can think of a tetrahedron in a usual manner, and show that "3-dimensional" affine space is the union of all lines through points of the "tetrahedron." In fact, for $r \in L$, the smallest affine element containing r is $\bigcup \{z \text{ in } L \mid z \leqslant l(xy) \text{ for some } x, y \leqslant r\}$. We use a result of Maeda [4] to show that, if r is in the affine lattice, then $M(r, x)$ holds in L for all x. We can also show that the affine lattice has the atomic exchange property by proving that, for r affine with $x \wedge r = 0$, then the join of x and r in the affine lattice covers r.

We use a result of MacLane's paper [3] to define affine independence and the invariance of the cardinality of a basis.

We can show the transitivity of parallelism, define parallel planes, and show that if s is a plane such that $s \wedge x = 0$ then $\exists$ a plane t such that $x \leqslant t$ and $s \parallel t$. Desargue's theorem also follows—both the planar and the non-planar cases.

At this point we can show that, for $x \in L$, $K_x = \{r \mid r \text{ affine and } x \leqslant r\}$ is an irreducible projective geometry. In fact axiom 5 can be weakened to postulate only three independent points, and the proof will go through. Using axiom 5 as originally stated it can be shown that, for x and y distinct, K_x and K_y are lattice isomorphic.

We now apply standard coordinatization techniques following the methods used in Artin [1] and obtain a coordinatization of our affine planes by an ordered division ring. In order to conclude that this ordered division ring is the real numbers, we need an Archimedean axiom and local compactness.

Let $y \leqslant x \vee z$, $y \neq x, z$. We say $(x \vee y) \sim (y \vee z)$ iff whenever $(p \vee q) \parallel (x \vee y)$ with $(p \vee x) \parallel (y \vee q)$ then $(p \vee y) \parallel (q \vee z)$.

AXIOM (7). Let $y \leqslant x \vee z$. Then $\exists\, x = y_0, y_1 = y, \ldots, y_n$ such that $(y_i \vee y_{i+1}) \sim (y_{i+1} \vee y_{i+2})$, $i = 0, \ldots, n-2$, and $z \leqslant y_{n-1} \vee y_n$. By taking $y = \tau(x)$ with τ a translation, we have Artin's version of the Archimedean axiom.

AXIOM (8). Suppose $\{x_\alpha, y_\alpha\}_{\alpha \in A}$ is a set such that $a \leqslant x \vee y \Rightarrow \exists\, \alpha \in A$ such that $a \leqslant x_\alpha \vee y_\alpha$, $a \neq x_\alpha, y_\alpha$. Suppose furthermore that $\forall\, d, e, f$, $\in \{x_\alpha, y_\alpha\}_{\alpha \in A}$, $\exists\, g, h$ such that $d \vee e \vee f \leqslant g \vee h$. Then $\exists\, B \subseteq A$, B finite such that $a \leqslant x \vee y \Rightarrow \exists\, \alpha \in B$ such that $a \leqslant x_\alpha \vee y_\alpha$, $a \neq x_\alpha, y_\alpha$.

The axioms (7) and (8) give us a locally compact Archimedean ordered division ring which makes our coordinatizing field be the reals.

Let us call a lattice satisfying axioms (1) to (8) a real convexity lattice. Using the invariance of the cardinality of a basis, we can next characterize the lattice L_n of convex subsets of a real n-dimensional vector space. A convex polyhedron in real n-space is an element of L_n which can be written as a finite join of atoms [2, Problem 29].

We can characterize modularity in a convexity lattice as follows. For $r, s \in L$ with $r \wedge s = 0$, $M(r, s)$ iff $r \wedge \bar{s} = 0$ where $\bar{s}$ is the affine closure of s. For, if $r \wedge \bar{s} = 0$ and $t \leqslant s$, $(t \vee r) \wedge s = \{b_i \mid b_i \leqslant c_i \vee a_i;$ $b_i \leqslant s,\ c_i \leqslant t$ and $a_i \leqslant r\}$. $b_i \nleqslant t \Rightarrow a_i \leqslant l(b_i c_i) \leqslant \bar{s}$, a contradiction. Thus we can show that $(t \vee r) \wedge s = t \vee (r \wedge s) = t$ and modularity obtains. Conversely, if $r \wedge \bar{s} \neq 0$, we take $x \leqslant r \wedge \bar{s}$. Then $x \leqslant l(b_1 b_2)$ for b_i distinct and under s. Since $r \wedge s = 0$, $x \nleqslant b_1 \vee b_2$. If $b_1 \leqslant x \vee b_2$, then $(b_2 \vee r) \wedge s \neq b_2$ but $b_2 \vee (r \wedge s) = b_2$ so (r, s) is not a modular pair.

For $r, s \in L$ with $r \wedge s \neq 0$, $M(r, s)$ iff for all $a_i \leqslant r$, $b_i, b_j \leqslant s$ with $b_j \leqslant a_i \vee b_i$, $b_i \neq b_j$; then $\exists\, d \leqslant r \wedge s$ such that $d \leqslant a_i \vee b_i$. If the condition holds and $t \leqslant s$, then $x \leqslant (t \vee r) \wedge s$, $x \leqslant c_i \vee a_i$ for c_i, a_i under t and r, respectively. Thus, if $x \neq c_i$, there exists $d \leqslant r \wedge s$ such that $d \leqslant a_i \vee c_i$. Either $d \leqslant x \vee c_i$ or $d \leqslant x \vee a_i$, and in both cases it is easy to show that $x \leqslant t \vee (r \wedge s)$. Thus $t \vee (r \wedge s) = (t \vee r) \wedge s$ and $M(r, s)$ holds. Conversely if $b_j \neq b_i$ both under s with $b_j \leqslant b_i \vee a_i$, $b_i \vee (r \wedge s) = (b_i \vee r) \wedge s$. Since $b_j \leqslant (b_i \vee r) \wedge s$, $b_j \leqslant b_i \vee d$ for some $d \leqslant r \wedge s$.

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